$$\chi_1^{\cdot} = \pm ((N + \varkappa G\rho)/(\rho F))^{1/2}$$

If, in addition, it is assumed that the load moves at uniform velocity and the solutions on the left and right of it are harmonic, two more critical velocities are defined

$$\chi_2^{\cdot} = \pm (E/\rho)^{1/2},\ \chi_3^{\cdot} = \pm ((N/(\rho F)) + (2\varkappa G/(3\rho)))^{1/2}$$

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ON THE LAW OF ANGULAR MOMENTUM VARIATION OF A SPHERE ROLLING ON A STATIONARY SURFACE*

A.S. SUMBATOV

The rolling of a homogeneous sphere without friction on a stationary surface is considered. The forms of surfaces are established, and the axes corresponding to them, relative to which the sphere angular momentum variation is determined, are defined by the same differential equation, as if the axes were stationary.

1. The theorem of the variation of the angular momentum $\mathbf{K}_A$ of a mechanical system relative to an arbitrary pole A has the form /1/

$$\mathbf{K}_A^{\cdot} + M\mathbf{v}_A \times \mathbf{v}_G = \sum \mathrm{mom}_A \mathbf{R} + \sum \mathrm{mom}_A \mathbf{F} \tag{1.1}$$

Here $\mathbf{v}_A$ is the velocity of the point A in fixed space, $M\mathbf{v}_G$ is the momentum of the body, and the right-hand side of (1.1), is the sum of the principal moments about the point A of the constraint reactions and the active forces operating on the system.

Suppose some axis AL constantly pass through the moving point A, and let $\mathbf{e}$ be the unit vector of that axis. If the constraints at each instant of time allow a virtual rotation of the system as a single rigid body about the AL axis and the kinematic condition

$$M(\mathbf{v}_G, \mathbf{v}_A \times \mathbf{e}) + (\mathbf{K}_A, \mathbf{e}^{\cdot}) = 0 \tag{1.2}$$

is satisfied /2/, then from (1.1) we have the scalar equation

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$$\dot{\mathbf{K}}_{AL} = \sum \mathrm{mom}_{AL}\,\mathbf{F} \tag{1.3}$$

which expresses the law of angular momentum variation of the system about the *AL* axis, as if that axis was stationary. The additional condition

$$\Sigma\,\mathrm{mom}_{AL}\mathbf{F} = 0$$

leads to the generalized integral of the areas

$$\mathbf{K}_{AL} = (\mathbf{K}_A, \mathbf{e}) = \mathrm{const}$$

More particular kinematic conditions than (1.2) are given in /1,3/.

Note that Eq.(1.2) is independent of the choice of the pole on the *AL* axis. Indeed, let *P* be an arbitrary point on the *AL* axis. We have

$$\mathbf{AP} = \sigma\mathbf{e}, \quad \mathbf{v}_P = \mathbf{v}_A + \sigma^{\cdot}\mathbf{e} + \sigma\mathbf{e}^{\cdot}, \quad \mathbf{K}_P = \mathbf{K}_A + M\mathbf{v}_G \times \sigma\mathbf{e}$$

and, consequently,

$$M(\mathbf{v}_G, \mathbf{v}_P \times \mathbf{e}) + (\mathbf{K}_P, \mathbf{e}^{\cdot}) = M(\mathbf{v}_G, \mathbf{v}_A \times \mathbf{e}) + (\mathbf{K}_A, \mathbf{e}^{\cdot})$$

To determine the variation of the angular momentum of the system in the form (1.3) it is necessary to select the respective pole and the direction of the axis. However, how this should be done in any specific case, is not known, since there is no general rule. Usually the stationary axis $(\mathbf{v}_A = \mathbf{e}^{\cdot} = 0)$, or the Koenig axis $(\text{т}.\,A \equiv \text{т}.\,G, \mathbf{e}^{\cdot} = 0)$. is taken as *AL*. Condition (1.2) is satisfied automatically, but not always among the virtual displacements is there a rotation of the system as a single rigid body about the chosen axis, for example, in the classical problem of the rolling a rigid body bounded by a regular convex surface on a stationary base. On the other hand, in this problem the constraints (no slip) allow virtual rotation of the body about an arbitrary axis *AL* drawn through the contact point *A* of the body with the supporting surface, and it is only necessary to determine the direction of that axis so that kinematic condition (1.2) is satisfied for all possible motions of the body.

We stress that for a given motion of the system, condition (1.2) may be considered as the differential equation with respect to $\mathbf{e}(t)$ which, obviously always has a solution. But the motion of the system is not known in advance, hence it is reasonable to assume that the unknown position of the pole and the direction of the movable axis are completely determined by constant parameters and the system configuration. In the problem of a rolling body this means that the unit vector of the *AL* axis is a function of, for instance, the Gaussian coordinates of the point *A* of the contacting surfaces and of the angle between the respective principal directions of these surfaces at the point *A*.

Below, we consider the problem of the rolling of a homogeneous sphere in which the selection of the moving direction in this formulation is fully analyzed. The problem of the integrability of the equations of rolling homogeneous spheres was investigated in /1,3-8/.

2. We will introduce the following notation: r is the geometric radius, ρ is the radius of inertia of the sphere, k_1 and k_2 are the principal curvatures, u, v are the Gaussian coordinates of the supporting surface, $AXYZ$ is the Darboux trihedron /9/ with origin at the point *A* of the sphere and supporting surface contact, k_{g1} and k_{g2} are the geodesic curvatures of the lines of curvature of the latter, $(v_1, v_2, 0)$ are the projections on the $AXYZ$ axes of the velocity of the pole A, φ is the angle of rotation of the sphere about the normal AZ to the Darboux trihedron, and (α, β, γ) are the components in $AXYZ$ axes of the unit vector $\mathbf{e}$ of the *AL* axis.

From the kinematic formulas for the rolling of one surface on another /1,7/, we have the following expressions in $AXYZ$ axes for the angular velocity components of the trihedron $AXYZ$

$$\Omega_1 = -k_2 v_2, \quad \Omega_2 = k_1 v_1, \quad \Omega_3 = k_{g1} v_1 + k_{g2} v_2$$

and for the angular velocity of the sphere

$$\omega_1 = -(k_2 + r^{-1})\, v_2, \quad \omega_2 = (k_1 + r^{-1})\, v_1, \quad \omega_3 = \varphi^{\cdot} + k_{g1} v_1 + k_{g2} v_2$$

From symmetry consideration it follows that $\mathbf{e} = \mathbf{e}(u, v)$. The absolute velocity of rotation of the *AL* axis is specified by the components

$$(\mathbf{e}^{\cdot})_1 = \left(\frac{\partial\alpha}{\partial u} + \gamma k_1 - \beta k_{g1}\right) v_1 + \left(\frac{\partial\alpha}{\partial v} - \beta k_{g2}\right) v_2$$

$$(\mathbf{e}^{\cdot})_2 = \left(\frac{\partial\beta}{\partial u} + \alpha k_{g1}\right) v_1 + \left(\frac{\partial\beta}{\partial v} + \gamma k_2 + \alpha k_{g2}\right) v_2$$

$$(\mathbf{e}^{\cdot})_3 = \left(\frac{\partial\gamma}{\partial u} - \alpha k_1\right) v_1 + \left(\frac{\partial\gamma}{\partial v} - \beta k_2\right) v_2$$

Having set up expression (1.2), we can equate to zero the quadratic form of the quantities $v_1, v_2, \varphi^{\cdot}$, that for any mutual position of sphere and support can take arbitrary values. Equating to zero the coefficients of that form we obtain the set of equations

$$\partial\beta/\partial u + \alpha k_{g1} = 0,\ \partial\alpha/\partial v - \beta k_{g2} = 0 \tag{2.1}$$

$$\partial\gamma/\partial u - \alpha k_1 = 0,\ \partial\gamma/\partial v - \beta k_2 = 0 \tag{2.2}$$

$$\rho^2 (k_2 - k_1)\gamma + (\rho^2 + r^2)\left[(rk_1 + 1)\left(\frac{\partial\beta}{\partial v} + \alpha k_{g2}\right) - (rk_2 + 1)\left(\frac{\partial\alpha}{\partial u} - \beta k_{g1}\right)\right] = 0 \tag{2.3}$$

which we supplement by two more equations

$$\alpha\left(\frac{\partial\alpha}{\partial u} - \beta k_{g1}\right) + \alpha\gamma k_1 = 0,\quad \beta\left(\frac{\partial\beta}{\partial v} + \alpha k_{g2}\right) + \beta\gamma k_2 = 0 \tag{2.4}$$

that are obtained by differentiating with respect to u and v the identity

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{2.5}$$

using Eqs.(2.1) and (2.2). The set of equations (2.1)-(2.5) is used to determine the unknown functions α, β, γ.

Eliminating in Eqs.(2.3) and (2.4) the quantities

$$\partial\alpha/\partial u - \beta k_{g1},\ \partial\beta/\partial v + \alpha k_{g2}$$

we obtain $\alpha\beta\gamma(k_2 - k_1) = 0$. If in the neighbourhood of point A of the supporting surface we have $k_2 = k_1$, then that neighbourhood is spherical /10/, and condition (1.2) is satisfied by the axis of constant direction /3/, otherwise $\alpha\beta\gamma = 0$. Let us analyze successively all possibilities that follow from this, assuming that the conditions in the cases enumerated below are satisfied in some neighbourhood of the point A on the supporting surface.

Let $\alpha = \beta = 0$. Then by Eq.(2.3), $k_2 = k_1$, the supporting surface is a sphere, and the AL axis is the common normal to the contacting spherical surfaces.

Let $\beta = \gamma = 0$(the case when $\alpha = \gamma = 0$ is similar). It follows from Eqs.(2.1)-(2.3) that $k_{g1} = k_{g2} = k_1 = 0$, and the supporting surface is arbitrarily cylindrical with the AL axis directed along its generatrix.

Let $\gamma = 0,\ \alpha\beta \not\equiv 0$. According to (2.2) we have $k_1 = k_2 = 0$, and the supporting surface is plane.

In these cases condition (1.2) can be satisfied by selecting the AL axis, which translates in stationary space, while the geometric properties of the supporting surface are in no way connected with the parameter of the sphere rolling on it. Let us examine the last possibility.

Let $\alpha = 0,\ \beta\gamma \not\equiv 0$ (the case when $\beta = 0,\ \alpha\gamma \not\equiv 0$ is similar). From Eqs.(2.1) and (2.2) and identity (2.5) it follows that

$$\beta = \cos f,\ \gamma = \sin f,\ f = f(v) \tag{2.6}$$

$$k_2 = f',\ k_{g2} = 0\quad (f' = df/dv) \tag{2.7}$$

The function $f(v)$ in (2.6) is not arbitrary, it is constrained by the requirement of kinematic feasibility of the sphere rolling without slip. For instance, when the sphere rolls on the inner side of a spherical surface it is obviously necessary that $f' > -r^{-1}$.

Equations (2.7) mean that the set of curvature lines $u = \text{const}$ consists of congruent curves which are geodesics on the supporting face, hence they are plane lines. Such a surface is called a cut /11/.

From Eq.(2.3), taking (2.6) and (2.7) into account, we have

$$k_1 = \frac{(\rho^2 + r^2)(rf' + 1)k_{g1}\operatorname{ctg} f - r^2 f'}{(\rho^2 + r^2)rf' + \rho^2} \tag{2.8}$$

Moreover the Liouville formula for the total curvature /11/

$$k_1 k_2 = \frac{\partial k_{g1}}{G\partial v} - \frac{\partial k_{g2}}{E\partial u} - (k_{g1})^2 - (k_{g2})^2$$

where $ds^2 = (Edu)^2 + (Gdv)^2$ is a linear element of the surface, yields

$$\frac{(\rho^2 + r^2)(rf' + 1)k_{g1}f'\operatorname{ctg} f - (rf')^2}{(\rho^2 + r^2)rf' + \rho^2} = \frac{\partial k_{g1}}{G\partial v} - (k_{g1})^2 \tag{2.9}$$

where $G = G(v)$ by virtue of the condition $k_{g2} = 0$.

Formulas (2.7)-(2.9) locally define the supporting surface geometry. The solutions of the form $k_{g1} = k_{g1}(v)$ of Eq.(2.9) are satisfied by some class of surfaces of revolution. Actually, in this case the lines $v = \text{const}$ have a constant curvature $\sqrt{(k_1)^2 + (k_{g1})^2}$, i.e. they are circles, and moreover are parallel, since the lines $u = \text{const}$ orthogonal to them are geodesics on the support surface.

Note also that the case

$$f = -\frac{\rho^2 v}{r(\rho^2 + r^2)} + \text{const},\quad k_{g1} = \frac{\rho^2}{r(\rho^2 + r^2)}\operatorname{tg} f$$

in which the right-hand side of (2.8) becomes indeterminate, corresponds to the rolling of the sphere on the inner side of a sphere of radius $r(\rho^2 + r^2)/\rho^2$.

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EVOLUTION OF A CONTACT DISCONTINUITY IN THE BAROTROPIC FLOW OF A VISCOUS GAS*

V.V. SHELUKHIN

Single-valued solvability as a whole is established with respect to time for an initial boundary value problem with discontinuity data for the equations of the one-dimensional barotropic flow of a viscous polytropic gas, and the behaviour of the solution is investigated, when the time increases without limit. The line of contact discontinuity is simulated by the trajectory of a piston of small mass located between two gases. In particular, if the discontinuity separates one and the same gas, it is shown that the pressure discontinuity can only disappear in an infinite time, and the discontinuity decays exponentially.

Suppose that at the initial instant $t=0$ the region $-1<\xi<0$ is filled with a gas of viscosity μ_1 with equation of state $p_1=a_1\rho^{\gamma_1}$, and the region $0<\xi<1$ is filled with a gas with corresponding characteristics μ_2 and $p_2=a_2\rho^{\gamma_2}$, where $\mu_i, a_i, \gamma_i>1$ $(i=1,2)$ are positive constants, p is the pressure and ρ is the density. Below, the velocity is denoted by u.

The behaviour of the medium in region $-1<\xi<1$ at $t>0$ is defined as follows. The motion of each gas outside the line of contact discontinuity $\xi=C(t), C(0)=0$ is defined by the equations

$$\rho(u_t+uu_\xi)=\mu u_{\xi\xi}-p_\xi,\ \rho_t+(\rho u)_\xi=0 \tag{1}$$

The conditions of contact discontinuity on the unknown line $\xi=C(t)$ have the form

$$[u]=[\mu u_\xi-p]=0,\ C'(t)=u\ \ ([u]=u(C(t)+0,t)-u(C(t)-0,t)) \tag{2}$$

Further, we will assume that at the points $\xi=-1, \xi=1$ the conditions of adhesion are satisfied

$$u(-1,t)=u(1,t)=0 \tag{3}$$

The functions $u_0(\xi), \rho_0(\xi)$,

$$u(\xi,0)=u_0(\xi),\ \rho(\xi,0)=\rho_0(\xi) \tag{4}$$

that specify the initial conditions are assumed to be smooth when $\xi\neq0$, while at the point $\xi=0$ the continuity of the functions ρ_0, p_0 is not required.

Problem (1)-(4) is conveniently solved in Lagrangian mass variables

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